

# RESCALING LIMITS IN NON-ARCHIMEDEAN DYNAMICS

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**ABSTRACT.** Suppose  $\{f_t\}$  is an analytic one-parameter family of rational maps defined over a non-Archimedean field  $K$ . We prove a finiteness theorem for the set of rescalings for  $\{f_t\}$ . This complements results of J. Kiwi.

## 1. INTRODUCTION

Let  $K$  be an algebraically closed field. For  $d \geq 1$ , let  $\text{Rat}_d(K)$  be the space of degree  $d$  rational maps over  $K$ , thought of as dynamical systems on  $\mathbb{P}^1(K)$ . The group  $\text{PGL}_2(K)$  acts on  $\text{Rat}_d(K)$  by conjugation. The moduli space of degree  $d$  rational maps on  $\mathbb{P}^1(K)$  is the quotient space  $\mathcal{M}_d(K) := \text{Rat}_d(K)/\text{PGL}_2(K)$ . Milnor [17] considered the moduli space  $\mathcal{M}_2(\mathbb{C})$  of quadratic complex rational maps and gave a dynamically natural compactification of  $\mathcal{M}_2(\mathbb{C})$ . Then using geometric invariant theory, Silverman [19, 20] compactified the moduli space  $\mathcal{M}_d(K)$  in general. DeMarco [8] also considered different compactifications of the moduli space  $\mathcal{M}_2(\mathbb{C})$ . To study the dynamics of complex rational maps approaching the boundary of  $\text{Rat}_d(\mathbb{C})$  (or  $\mathcal{M}_d(\mathbb{C})$ ), Kiwi [16] considered rescaling limits for a holomorphic family  $\{f_t\}$  (resp. a sequence  $\{f_n\}$ ) in  $\text{Rat}_d(\mathbb{C})$ . These arise as limits  $M_t^{-1} \circ f_t^q \circ M_t \rightarrow g$  (resp.  $M_n^{-1} \circ f_n^q \circ M_n \rightarrow g$ ) of rescaled iterates where the convergence is locally uniform outside some finite subset of  $\mathbb{P}^1(\mathbb{C})$ . By regarding a holomorphic family as a rational map with coefficients in the field of formal Puiseux series, and by studying its induced action on the corresponding Berkovich space, Kiwi proved for any given holomorphic one-parameter family of degree  $d \geq 2$  rational maps, there are at most  $2d - 2$  dynamically independent rescalings such that the corresponding rescaling limits are not postcritically finite [16, Theorem 1, Theorem 2]. Later, Arfeux [1] announced the same results using the Deligne-Mumford compactification of the moduli space of the stable punctured spheres.

An algebraically closed complete valued field is isomorphic to either the field of complex numbers  $\mathbb{C}$  or a non-Archimedean field [9]. Our main result translates Kiwi's finiteness result to non-Archimedean algebraically closed fields, subject to a natural tameness hypothesis. We now set up the statement.

Throughout this paper,  $K$  will denote an algebraically closed field which is complete with respect to a nontrivial non-Archimedean absolute value  $|\cdot|_K$ . Let  $\phi(z) \in K(z)$  be a rational map. We can write  $\phi = \phi_1 \circ \phi_2$ , where  $\phi_1$  is a separable rational map and  $\phi_2(z) = z^{p^j}$  for some  $j \geq 0$  if the field  $K$  has positive characteristic  $p > 0$  or  $\phi_2(z) = z$  if the field  $K$  has characteristic zero. The rational map  $\phi_1$  is called the separable part of  $\phi$ . The degree of  $\phi_1$  is called the nontrivial degree of  $\phi$  and the preimages of critical points of  $\phi_1$  under  $\phi_2$  are called the nontrivial critical points of  $\phi$ . We say rational map  $\phi \in K(z)$  is postcritically finite at

nontrivial critical points if each nontrivial critical point of  $\phi$  has a finite forward orbit; equivalently, each critical value of  $\phi_1$  has a finite forward orbit under  $\phi$ .

As in Kiwi's study of degenerating rational maps over  $\mathbb{C}$ , we now study families  $\phi_t$  approaching boundary of  $\text{Rat}_d(K)$  (or  $\mathcal{M}_d(K)$ ). Since the field  $K$  is not locally compact with respect to the absolute value  $|\cdot|_K$ , the definition of rescaling limits in [16] needs to be slightly modified. The non-Archimedean property turns out to make pointwise convergence suitable; see Definition 2.4 and Proposition 3.5. For instance, let  $f_t(z) = \frac{z^3+t}{z} \in K(z)$ , then, as  $t \rightarrow 0$ ,  $f_t(z)$  converges to  $z^2$  pointwise on  $\mathbb{P}^1(K) \setminus \{0\}$ , but  $f_t(0) = \infty$  for all  $t \neq 0$ . For an analytic family  $\{f_t\} \subset \text{Rat}_d(K)$ , we can associate to  $\{f_t\}$  a rational map  $\mathbf{f} : \mathbb{P}^1(\mathbb{L}) \rightarrow \mathbb{P}^1(\mathbb{L})$ , where  $\mathbb{L}$  is the completion of the formal Puiseux series over  $K$ . For Puiseux series and the field  $\mathbb{L}$ , we refer [9, 14, 15]. The space  $\mathbb{P}^1(\mathbb{L})$  is naturally a subset of the corresponding Berkovich space  $\mathbb{P}_{\text{Ber}}^1(\mathbb{L})$ ; see [3, 13] for details. The rational map  $\mathbf{f}$  induces a map on  $\mathbb{P}_{\text{Ber}}^1(\mathbb{L})$  extending its action on  $\mathbb{P}^1(\mathbb{L})$ , so we also use notation  $\mathbf{f}$  for the induced map. Let  $\mathcal{R}_{\mathbf{f}}$  be the Berkovich ramification locus, that is the set of points in  $\mathbb{P}_{\text{Ber}}^1(\mathbb{L})$  such that the local degrees of  $\mathbf{f}$  at these points are at least 2, i.e.  $\mathcal{R}_{\mathbf{f}} = \{\xi \in \mathbb{P}_{\text{Ber}}^1(\mathbb{L}) : \deg_{\xi} \mathbf{f} \geq 2\}$ . It is a closed subset of  $\mathbb{P}_{\text{Ber}}^1(\mathbb{L})$  with no isolated points and has at most  $\deg \mathbf{f} - 1$  connected components [10, Theorem A], each of which has tree structure. Following [21], we say the rational map  $\mathbf{f} : \mathbb{P}_{\text{Ber}}^1(\mathbb{L}) \rightarrow \mathbb{P}_{\text{Ber}}^1(\mathbb{L})$  is tame if  $\mathcal{R}_{\mathbf{f}}$  has only finitely many points with valence at least 3 in  $\mathcal{R}_{\mathbf{f}}$ .

We will prove

**Theorem 1.1.** *Let  $\{f_t(z)\} \subset K(z)$  be an analytic family of rational maps of non-trivial degree  $d \geq 2$  and let  $\mathbf{f}_1$  be the separable part of the associated rational map  $\mathbf{f}$  of  $\{f_t\}$ . Assume  $\mathbf{f}_1$  is tame. Then there are at most  $2d - 2$  pairwise dynamically independent rescalings for  $\{f_t\}$  such that the corresponding rescaling limits are not postcritically finite at nontrivial critical points.*

In section 6, we give some examples of analytic families with rescaling limits that are not postcritically finite at nontrivial critical points.

The tameness hypothesis of the separable part  $\mathbf{f}_1$  is needed in order to prove a non-Archimedean Rolle's theorem in positive characteristic, see Lemma 5.1. If  $K$  has characteristic zero or positive characteristic  $p > \deg \mathbf{f}_1$ , then the rational map  $\mathbf{f}_1 : \mathbb{P}_{\text{Ber}}^1(\mathbb{L}) \rightarrow \mathbb{P}_{\text{Ber}}^1(\mathbb{L})$  is tame [10, Corollary 6.6]. Thus

**Corollary 1.2.** *Suppose the field  $K$  has characteristic zero. Let  $\{f_t(z)\} \subset K(z)$  be an analytic family of degree  $d \geq 2$  rational maps. Then there are at most  $2d - 2$  pairwise dynamically independent rescalings for  $\{f_t\}$  such that the corresponding rescaling limits are not postcritically finite.*

**Outline.** In section 2, we recall the relevant backgrounds of Berkovich space and define the rescaling limits for an analytic family of rational maps over  $K$ . The goal of section 3 is to discuss the reduction map and show the relations between reductions and rescaling limits. Section 4 is devoted to restating Kiwi's results which are still true for the case when  $K$  has characteristic zero. Finally, we prove Theorem 1.1 in section 5 and give examples to illustrate it in section 6.

## 2. PRELIMINARIES

**2.1. Non-Archimedean fields.** For the field  $K$ , let  $|K^\times|_K \subset (0, \infty)$  be the set of absolute values attained by nonzero elements of  $K$ , which is called the value group

of  $K$ . Then  $|K^\times|_K$  is dense in  $(0, \infty)$  since  $K$  is algebraically closed, and hence  $K$  can not be locally compact. Let  $\mathcal{O}_K = \{z \in K : |z|_K \leq 1\}$  be the ring of integers of  $K$  and let  $\mathcal{M}_K = \{z \in K : |z|_K < 1\}$  be the unique maximal ideal of  $\mathcal{O}_K$ . Let  $k = \mathcal{O}_K/\mathcal{M}_K$  be the residue field. Note if  $\text{char } K = p > 0$  then  $\text{char } k = p$ , but if  $\text{char } K = 0$ , then  $k$  could have any characteristic. For instance, for a prime number  $p \geq 2$ , if  $K$  is the completion of the field of formal Puiseux series over  $\mathbb{F}_p$  with its natural absolute value, then  $\text{char } K = \text{char } k = p$ ; if  $K$  is the complex  $p$ -adic field  $\mathbb{C}_p$ , then  $k = \overline{\mathbb{F}_p}$ , the algebraic closure of  $\mathbb{F}_p$ , and  $\text{char } K = 0$  but  $\text{char } k = p$ .

Given  $a \in K$  and  $r > 0$ , define

$$D(a, r) := \{z \in K : |z - a|_K < r\} \quad \text{and} \quad \overline{D}(a, r) := \{z \in K : |z - a|_K \leq r\}.$$

If  $r \in |K^\times|_K$ , we say that  $D(a, r)$  is an open rational disk in  $K$  and  $\overline{D}(a, r)$  is a closed rational disk in  $K$ . If  $r \notin |K^\times|_K$ , we call  $D(a, r) = \overline{D}(a, r)$  an irrational disk. Let  $U(a, r) \subset K$  be a disk centered at  $a \in K$  with radius  $r > 0$ , that is,  $U$  has the form  $D(a, r)$  or  $\overline{D}(a, r)$ . Then if  $b \in U(a, r)$ , we have  $U(a, r) = U(b, r)$ . Moreover, the radius  $r$  is the same as the diameter of  $U(a, r)$ , that is  $r = \sup\{|z - w|_K : z, w \in U(a, r)\}$ . Furthermore, if two disks have a nonempty intersection, then one must contain the other. Finally, we should mention here every disk in  $K$  is both open and closed under the topology of  $K$ .

Let  $K\{\{t\}\}$  be the field of formal Puiseux series over  $K$ . Then  $K\{\{t\}\}$  can be equipped with a non-Archimedean absolute value  $|\cdot|_{K\{\{t\}\}}$  by fixing a number  $\epsilon \in (0, 1)$  and defining  $|\sum_{n \in \mathbb{Z}} a_n t^{\frac{n}{m}}|_{K\{\{t\}\}} = \epsilon^{\frac{n_0}{m}}$ , where  $n_0$  is the smallest integer  $n \in \mathbb{Z}$  such that  $a_n \neq 0$ . Then the field  $K\{\{t\}\}$  is algebraically closed but not complete. Let  $\mathbb{L}$  be the completion of the field  $K\{\{t\}\}$ . It consists of all formal sums of the form  $\sum_{n \geq 0} a_n t^{q_n}$ , where  $\{q_n\}$  is a sequence of rational numbers increasing to  $\infty$  and  $a_n \in K$ . Let  $|\cdot|_{\mathbb{L}}$  be the extension of the non-Archimedean absolute value  $|\cdot|_{K\{\{t\}\}}$ . Then the ring of integer of the field  $\mathbb{L}$  is

$$\mathcal{O}_{\mathbb{L}} = \{z \in \mathbb{L} : |z|_{\mathbb{L}} \leq 1\} = \left\{ \sum_{n \geq 0} a_n t^{q_n} : q_n \geq 0 \right\}$$

and the unique maximal ideal  $\mathcal{M}_{\mathbb{L}}$  of  $\mathcal{O}_{\mathbb{L}}$  consists of series with zero constant term, i.e.

$$\mathcal{M}_{\mathbb{L}} = \{z \in \mathbb{L} : |z|_{\mathbb{L}} < 1\} = \left\{ \sum_{n \geq 0} a_n t^{q_n} : q_n > 0 \right\}.$$

The residue field  $\mathcal{O}_{\mathbb{L}}/\mathcal{M}_{\mathbb{L}}$  is canonically isomorphic to  $K$ .

**2.2. The Berkovich projective line.** In this subsection, we summarize some fundamental properties of the Berkovich projective line, for details we refer [3, 4, 6].

The Berkovich affine line  $\mathbb{A}_{Ber}^1(K)$  is the set of all multiplicative seminorms on the ring  $K[z]$  of polynomials over  $K$ , whose restriction to the field  $K \subset K[z]$  is equal to the given absolute value  $|\cdot|_K$ . For  $a \in K$  and  $r \geq 0$ , let  $\xi_{a,r}$  be the seminorm defined by  $|f|_{\xi_{a,r}} = \sup_{z \in \overline{D}(a,r)} |f(z)|_K$ . Then there are 4 types of points in

$\mathbb{A}_{Ber}^1(K)$ :

1. Type I.  $\xi_{a,0}$  for some  $a \in K$ .
2. Type II.  $\xi_{a,r}$  for some  $a \in K$  and  $r \in |K^\times|$ .
3. Type III.  $\xi_{a,r}$  for some  $a \in K$  and  $r \notin |K^\times|$ .

4. Type IV. A limit of seminorms  $\{\xi_{a_i, r_i}\}_{i \geq 0}$ , where the corresponding sequence of closed disks  $\{\overline{D}_{a_i, r_i}\}_{i \geq 0}$  satisfies  $\overline{D}_{a_{i+1}, r_{i+1}} \subset \overline{D}_{a_i, r_i}$  and  $\bigcap_i \overline{D}_{a_i, r_i} = \emptyset$ .

We can identify  $K$  with the type I points in  $\mathbb{A}_{Ber}^1(K)$  via  $a \rightarrow \xi_{a,0}$ . The point  $\xi_{0,1} \in \mathbb{A}_{Ber}^1(K)$  is called the Gauss point and denoted by  $\xi_G$ . We put the Gelfand topology (weak topology) on  $\mathbb{A}_{Ber}^1(K)$ , which makes the map  $\mathbb{A}_{Ber}^1(K) \rightarrow [0, +\infty)$  sending  $\xi$  to  $|f|_\xi$  continuous for each  $f \in K[z]$ . Then  $\mathbb{A}_{Ber}^1(K)$  is locally compact, Hausdorff and uniquely path-connected.

The Berkovich projective line  $\mathbb{P}_{Ber}^1(K)$  is obtained by gluing two copies of  $\mathbb{A}_{Ber}^1(K)$  along  $\mathbb{A}_{Ber}^1(K) \setminus \{0\}$  via the map  $\xi \rightarrow \frac{1}{\xi}$ . Then we can associate the Gelfand topology on  $\mathbb{P}_{Ber}^1(K)$ . The Berkovich projective line  $\mathbb{P}_{Ber}^1(K)$  is a compact, Hausdorff, uniquely path-connected topological space and contains  $\mathbb{P}^1(K)$  as a dense subset.

The space  $\mathbb{P}_{Ber}^1(K)$  has tree structure. For a point  $\xi \in \mathbb{P}_{Ber}^1(K)$ , we can define an equivalence relation on  $\mathbb{P}_{Ber}^1(K) \setminus \{\xi\}$ , that is,  $\xi'$  is equivalent to  $\xi''$  if  $\xi'$  and  $\xi''$  are in the same connected component of  $\mathbb{P}_{Ber}^1(K) \setminus \{\xi\}$ . Such an equivalence class  $\vec{v}$  is called a direction at  $\xi$ . We say that the set  $T_\xi \mathbb{P}_{Ber}^1(K)$  formed by all directions at  $\xi$  is the tangent space at  $\xi$ . For  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(K)$ , denote by  $\mathbf{B}_\xi(\vec{v})^-$  the component of  $\mathbb{P}_{Ber}^1(K) \setminus \{\xi\}$  corresponding to the direction  $\vec{v}$ . If  $\xi \in \mathbb{P}_{Ber}^1(K)$  is a type I or IV point,  $T_\xi \mathbb{P}_{Ber}^1(K)$  consists of a single direction. If  $\xi \in \mathbb{P}_{Ber}^1(K)$  is a type II point, the directions in  $T_\xi \mathbb{P}_{Ber}^1(K)$  are in one-to-one correspondence with the elements in  $\mathbb{P}^1(k)$ . If  $\xi \in \mathbb{P}_{Ber}^1(K)$  is a type III point,  $T_\xi \mathbb{P}_{Ber}^1(K)$  consists of two directions. Note the Gauss point  $\xi_G$  is a type II point. We can identify  $T_{\xi_G} \mathbb{P}_{Ber}^1(K)$  to  $\mathbb{P}^1(k)$  by the correspondence  $T_{\xi_G} \mathbb{P}_{Ber}^1(K) \rightarrow \mathbb{P}^1(k)$  sending  $\vec{v}_x$  to  $x$ , where  $\vec{v}_x$  is the direction at  $\xi_G$  such that  $\mathbf{B}_{\xi_G}(\vec{v}_x)^-$  contains all the type I points whose images are  $x$  under the canonical reduction map  $\mathbb{P}^1(K) \rightarrow \mathbb{P}^1(k)$ .

**2.3. Rational maps.** In this subsection, we consider rational maps over the field  $K$  and define an analytic family of rational maps over  $K$ . For rational maps over a non-Archimedean field, we refer [3–5].

We first define analytic maps on a disk  $U \subset K$ .

**Definition 2.1.** Let  $U \subset K$  be a disk and  $z_0 \in U$ . We say a map  $f : U \rightarrow K$  is analytic if  $f$  can be written as a power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \in K[[z - z_0]],$$

which converges for all  $z \in U$ . The smallest  $n$  such that  $c_n \neq 0$  is called the order of  $f$  at  $z_0$  and denoted  $\text{ord}_{z_0}(f)$ .

It is easy to check that analytic property is independent of the choice of  $z_0 \in U$ . Moreover, if  $U = \overline{D}(z_0, r)$  is a rational closed disk, then  $\sum_{n=0}^{\infty} c_n (z - z_0)^n$  converges for each  $z \in U$  if and only if  $\lim_{n \rightarrow \infty} |c_n|_K r^n = 0$ . For rational open and irrational disks,  $\lim_{n \rightarrow \infty} |c_n|_K r^n = 0$  implies  $\sum_{n=0}^{\infty} c_n (z - z_0)^n$  converges, but the converse is not true.

We denote by  $\mathbb{P}^1(K) := K \cup \{\infty\}$  the projective line over  $K$ . We define the spherical metric on  $\mathbb{P}^1(K)$  as follows: for points  $z = [x : y]$  and  $w = [u : v]$  in

$\mathbb{P}^1(K)$ ,

$$\Delta(z, w) := \frac{|xv - yu|_K}{\max\{|x|_K, |y|_K\} \max\{|u|_K, |v|_K\}}.$$

Equivalently,

$$\Delta(z, w) := \begin{cases} \frac{|z-w|_K}{\max\{1, |z|_K\} \max\{1, |w|_K\}}, & \text{if } z, w \in K, \\ \frac{1}{\max\{1, |z|_K\}}, & \text{if } z \in K, w = \infty. \end{cases}$$

Recall a degree  $d \geq 1$  rational map  $f : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$  is represented by a pair  $f_1, f_2 \in K[X, Y]$  of degree  $d$  homogeneous polynomials with no common factors, that is,  $f([X : Y]) = [f_1(X, Y) : f_2(X, Y)]$  for all  $[X : Y] \in \mathbb{P}^1(K)$ . Equivalently, the map  $f$  can be considered as the quotient of two relatively prime polynomials, of which the greatest degree is  $d$ . Let  $\text{Rat}_d(K)$  denote the set of rational maps of degree  $d$  over  $K$ . Then  $\text{Rat}_d(K)$  can be naturally identified with an open subset of  $\mathbb{P}^{2d+1}(K)$  via the map  $\text{Rat}_d(K) \rightarrow \mathbb{P}^{2d+1}(K)$  sending  $\frac{a_d z^d + \dots + a_0}{b_d z^d + \dots + b_0}$  to  $[a_d : \dots : a_0 : b_d : \dots : b_0]$ .

Let  $\phi(z) \in K(z)$  be a rational map. Suppose  $z_0 \in \mathbb{P}^1(K)$  and set  $w_0 = \phi(z_0)$ . Pick  $\psi_1, \psi_2 \in \text{PGL}_2(K)$  such that  $\psi_1(0) = z_0$  and  $\psi_2(w_0) = 0$ , and define  $\Phi = \psi_2 \circ \phi \circ \psi_1$ . The multiplicity  $m_\phi(z_0)$  of  $\phi$  at  $z_0$  is the order of  $\Phi$  at 0. The weight  $w_\phi(z_0)$  of  $\phi$  at  $z_0$  is the order of  $\Phi'$  at 0. If  $\Phi'(z) \equiv 0$ , we set  $w_\phi(z_0) = \infty$ . This can happen: for example, if  $\text{char } K = p$  and  $\phi(z) = z^p$ , then  $\phi'(z) = 0$  for each  $z \in K$ . A point  $z_0 \in \mathbb{P}^1(K)$  is called a critical point of  $\phi$  if  $w_\phi(z_0) > 0$ . Denote  $\text{Crit}(\phi) \subset \mathbb{P}^1(K)$  for the set of all critical points of  $\phi$ . If every point  $z \in \mathbb{P}^1(K)$  is a critical point of  $\phi$ , then we say  $\phi$  is inseparable. Otherwise,  $\phi$  is called separable. Recall that for every rational map  $\phi \in K(z)$ , we can write  $\phi(z) = \phi_1 \circ \phi_2(z)$ , where  $\phi_1$  is a separable rational map and  $\phi_2(z) = z^{p^j}$  for some  $j \geq 0$  if the field  $K$  has positive characteristic  $p > 0$  or  $\phi_2(z) = z$  if the field  $K$  has characteristic zero. It is called the (in)separable decomposition of  $\phi$ . The rational map  $\phi_1$  is called the separable part of  $\phi$ . We define the nontrivial degree  $\deg_0 \phi := \deg \phi_1$  and the nontrivial critical set  $\text{Crit}_0(\phi) := \phi_2^{-1}(\text{Crit}(\phi_1))$  of  $\phi$ .

**Definition 2.2.** Let  $U \subset K$  be a disk containing 0. A collection  $\{f_t\}_{t \in U} \subset \mathbb{P}^{2d+1}(K)$  is a 1-dimensional separable analytic family of degree  $d \geq 1$  rational maps if the map  $F : U \rightarrow \mathbb{P}^{2d+1}(K)$  sending  $t$  to  $f_t$  is an analytic map such that  $f_t \in \text{Rat}_d(K)$  is separable for all  $t \neq 0$ . If  $\{f_t\}_{t \in U}$  is a 1-dimensional separable analytic family of degree 1 rational maps, we call it a moving frame.

Let  $U \subset K$  be a disk containing 0. We say  $\{f_t\}_{t \in U} \subset \mathbb{P}^{2d+1}(K)$  is a 1-dimensional analytic family of nontrivial degree  $d' \geq 1$  rational maps if we can write  $f_t = g_t \circ h$ , where  $\{g_t\}_{t \in U} \subset \mathbb{P}^{2d'+1}(K)$  is a 1-dimensional separable analytic family of degree  $d' \geq 1$  rational maps and  $h(z) = z^{p^j}$  for some  $j \geq 0$  if  $\text{Char } K = p > 0$  or  $\phi_2(z) = z$  if  $\text{Char } K = 0$ .

**Remark 2.3.**

(1) We are really interested in the germ defined by an analytic family, so considering a small disk  $V \subset U$  containing 0 if necessary, we can always assume  $U = \overline{D}(0, r)$  is a rational closed disk.

(2) For an analytic family  $\{f_t\}$  on  $U$ , we can write

$$f_t(z) = \frac{P(z)}{Q(z)} = \frac{a_d(t)z^d + \dots + a_0(t)}{b_d(t)z^d + \dots + b_0(t)}$$

and denote by  $\ell$  the minimum among the orders of the  $a_i(t)$  and  $b_j(t)$ , at the origin,  $i, j = 1, \dots, d$ . Let

$$C = \max \left\{ 1, \left| \lim_{t \rightarrow 0} \frac{a_d(t)}{t^\ell} \right|_K, \dots, \left| \lim_{t \rightarrow 0} \frac{a_0(t)}{t^\ell} \right|_K, \left| \lim_{t \rightarrow 0} \frac{b_d(t)}{t^\ell} \right|_K, \dots, \left| \lim_{t \rightarrow 0} \frac{b_0(t)}{t^\ell} \right|_K \right\}$$

Let  $x \in K$  be an element such that  $|x|_K = C$ . For  $t$  sufficiently small, by considering  $g_t(z) = \frac{P(z)/xt^\ell}{Q(z)/xt^\ell}$  if necessary, we can assume  $\{f_t\} \subset \mathcal{O}_K(z)$  and that  $f_t$  has at least one coefficient with absolute value 1. Therefore, throughout this paper, for a rational map  $\phi(z) \in K(z)$ , we assume  $\phi(z) \in \mathcal{O}_K(z)$  and at least one coefficient has absolute value 1.

For an analytic family  $\left\{ f_t = \frac{a_d(t)z^d + \dots + a_0(t)}{b_d(t)z^d + \dots + b_0(t)} \right\} \subset K(z)$  of degree  $d$  rational maps, let  $\mathbf{a}_d, \dots, \mathbf{a}_0, \mathbf{b}_d, \dots, \mathbf{b}_0$  be the power series expressions of the coefficients  $a_d(t), \dots, a_0(t), b_d(t), \dots, b_0(t)$ , respectively. Then the degree  $d$  rational map  $\mathbf{f} : \mathbb{P}^1(\mathbb{L}) \rightarrow \mathbb{P}^1(\mathbb{L})$  given by

$$\mathbf{f}(z) = \frac{\mathbf{a}_d z^d + \dots + \mathbf{a}_0}{\mathbf{b}_d z^d + \dots + \mathbf{b}_0}$$

is called, following Kiwi, the rational map associated to  $\{f_t\}$ . The rational map  $\mathbf{f}$  induces a map from  $\mathbb{P}_{Ber}^1(\mathbb{L})$  to itself. We use the same notation  $\mathbf{f}$  for the induced map. Then for  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L})$ ,  $\mathbf{f}(\xi)$  is the unique point in  $\mathbb{P}_{Ber}^1(\mathbb{L})$  such that  $|g|_{\mathbf{f}(\xi)} = |\mathbf{f} \circ g|_\xi$  for all  $g \in \mathbb{L}(z)$ .

**2.4. Rescaling limits for an analytic family.** A non-Archimedean field is locally compact if and only if it is discretely valued and has finite residue field [7]. Then  $K$  is not locally compact, hence neither is  $\mathbb{P}^1(K)$ . Thus, we define the rescaling limits for an analytic family of rational maps over  $K$  in the following sense:

**Definition 2.4.** Let  $\{f_t\}$  be an analytic family of rational maps of nontrivial degree at least 2. A moving frame  $\{M_t\}$  is called a rescaling for  $\{f_t\}$  if there exist an integer  $q \geq 1$ , a rational map  $g : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$  of nontrivial degree  $d' \geq 2$  and a finite subset  $S$  of  $\mathbb{P}^1(K)$  such that, as  $t \rightarrow 0$ ,

$$M_t^{-1} \circ f_t^q \circ M_t(z) \rightarrow g(z)$$

pointwise on  $\mathbb{P}^1(K) \setminus S$ . We say  $g$  is a rescaling limit for  $\{f_t\}$  in  $\mathbb{P}^1(K) \setminus S$ . The minimal  $q \geq 1$  such that the above holds is called the period of the rescaling  $\{M_t\}$ .

Following [16], we define the following equivalence relations on the set of all rescalings.

**Definition 2.5.** Two moving frames  $\{M_t\}$  and  $\{L_t\}$  are equivalent if there exists  $M \in \text{Rat}_1(K)$  such that  $M_t^{-1} \circ L_t \rightarrow M$  as  $t \rightarrow 0$ .

**Definition 2.6.** Two rescalings  $\{M_t\}$  and  $\{L_t\}$  for an analytic family  $\{f_t\}$  are dynamically dependent if there exist an integer  $l \geq 0$  and a nonconstant rational map  $g$  such that  $L_t^{-1} \circ f^l \circ M_t \rightarrow g$ , as  $t \rightarrow 0$ , pointwise outside some finite set.

If  $\{M_t\}$  and  $\{L_t\}$  are two equivalent rescalings for an analytic family  $\{f_t\}$ , then they are dynamically dependent. The converse is not true in general.

### 3. REDUCTION

Recall  $K$  is any arbitrary complete algebraically closed non-Archimedean field. Let  $g \in \mathcal{O}_K(z)$  be a rational map. Then reducing the coefficients of  $g$  modulo  $\mathcal{M}_K$  and canceling common factors, we get a rational map  $\tilde{g}$  over the residue field  $k$ , which is called the reduction of  $g$ . Now we can define a map

$$\rho_K : \text{Rat}_d(K) \rightarrow \text{Rat}_{\leq d}(k),$$

where  $\text{Rat}_{\leq d}(k)$  is the space of degree at most  $d$  rational maps over  $k$ , sending  $g$  to its reduction  $\tilde{g}$ . We call  $\rho_K$  the reduction map for rational maps over  $K$ .

We first state an easy proposition and omit the proof.

**Proposition 3.1.** *Let  $\phi(z), \psi(z) \in K(z)$  be rational maps, and let  $\rho(\phi)$  and  $\rho(\psi)$  be their reductions, respectively. Then*

- (1)  $\rho(\phi \cdot \psi) = \rho(\phi) \cdot \rho(\psi)$ ,
- (2)  $\rho(\phi + \psi) = \rho(\phi) + \rho(\psi)$ ,
- (3) *If  $\deg \rho(\psi) \geq 1$ , then  $\rho(\phi \circ \psi) = \rho(\phi) \circ \rho(\psi)$ .*

In Proposition 3.1 (3), if  $\deg \rho(\psi) = 0$ , the situation is complicated. For example, let  $K$  be the completion of the formal Puiseux series over  $\mathbb{C}$  and define rational maps  $\phi(z) = \frac{z^2}{t^2}$  and  $\psi(z) = tz^2$  over  $K$ . Then  $\rho(\phi \circ \psi)(z) = z^4$  but  $(\rho(\phi) \circ \rho(\psi))(z) = \infty$  since  $\rho(\phi) = \infty$ .

Recall  $\mathbb{L}$  is the completion of the field of formal Puiseux series over  $K$ . Since  $\mathbb{L}$  is an algebraically closed and complete non-Archimedean field, we can consider the reduction map  $\rho_{\mathbb{L}} : \text{Rat}_d(\mathbb{L}) \rightarrow \text{Rat}_{\leq d}(K)$  of rational maps over  $\mathbb{L}$ .

**Definition 3.2.** *Let  $\{f_t\}$  be an analytic family of degree  $d \geq 1$  rational maps. We say  $\{f_t\}$  has good reduction if the associated rational map  $\mathbf{f}$  has good reduction, that is,  $\deg \rho_{\mathbb{L}}(\mathbf{f}) = d$ . Otherwise, we say  $\{f_t\}$  has bad reduction. If there is a moving frame  $\{M_t\} \subset \mathbb{P}^3(K)$  such that  $\{M_t^{-1} \circ f_t \circ M_t\}$  has good reduction, we say that  $\{f_t\}$  has potentially good reduction.*

Given  $f = [f_1 : f_2] \in \mathbb{P}^{2d+1}(K)$ , we can write

$$f = [f_1 : f_2] = [H_f \hat{f}_1 : H_f \hat{f}_2] = H_f [\hat{f}_1 : \hat{f}_2] = H_f \hat{f},$$

where  $H_f = \gcd(f_1, f_2)$  is a homogeneous polynomial and  $\hat{f} = [\hat{f}_1 : \hat{f}_2]$  is a rational map of degree at most  $d$ .

**Proposition 3.3.** *Suppose  $\{f_t\}$  is an analytic family of degree  $d \geq 1$  rational maps such that  $f_t \rightarrow H_f \hat{f}$ , as  $t \rightarrow 0$ , in  $\mathbb{P}^{2d+1}(K)$ . Then, as  $t \rightarrow 0$ ,  $f_t$  converges to  $\hat{f}$  pointwise on  $\mathbb{P}^1(K) \setminus \{H_f = 0\}$ .*

*Proof.* Write  $f_t(z) = \frac{P_t(z)}{Q_t(z)}$  and  $f(z) = H_f \hat{f}(z) = \frac{H_f(z)P(z)}{H_f(z)Q(z)}$ . Fix  $z \in \mathbb{P}^1(K) \setminus \{H_f = 0\}$ .

(1) If  $\hat{f}(z) \neq \infty$ , then  $f_t(z) \neq \infty$  for  $t$  sufficiently small. In fact, if there exists  $\{t_i\}$  such that, as  $i \rightarrow \infty$ ,  $t_i \rightarrow 0$  and  $f_{t_i}(z) = \infty$ , then either  $z = \infty \notin \{H_f = 0\}$  and  $\deg P_{t_i} > \deg Q_{t_i}$ , which implies  $\deg P > \deg Q$ , or  $Q_{t_i}(z) = 0$ . Note as  $t \rightarrow 0$ ,

$$|Q_{t_i}(z) - H_f(z)Q(z)|_K \leq \max\{|c_n(t)z^n|_K : 0 \leq n \leq d\} \rightarrow 0,$$

where  $c_n(t)$  is the coefficient of  $z^n$  in the polynomial  $Q_{t_i}(z) - H_f Q(z)$ . Thus, if  $Q_{t_i}(z) = 0$ , we have  $Q(z) = 0$  since  $H_f(z) \neq 0$ .

Thus, we have

$$\begin{aligned}
\Delta(f_t(z), \hat{f}(z)) &= \frac{|f_t(z) - \hat{f}(z)|_K}{\max\{1, |f_t(z)|_K\} \max\{1, |\hat{f}(z)|_K\}} \\
&\leq |f_t(z) - \hat{f}(z)|_K \\
&= \left| \frac{P_t(z)}{Q_t(z)} - \frac{P(z)}{Q(z)} \right|_K \\
&= \left| \frac{P_t(z)}{Q_t(z)} - \frac{H(z)P(z)}{H(z)Q(z)} \right|_K \\
&= \frac{|P_t(z)H(z)Q(z) - Q_t(z)H(z)P(z)|_K}{|H(z)Q(z)Q_t(z)|_K} \\
&\rightarrow 0
\end{aligned}$$

because

$$|P_t(z)H(z)Q(z) - Q_t(z)H(z)P(z)|_K \leq \max\{|\tilde{c}_n(t)z^n|_K : 0 \leq n \leq d\} \rightarrow 0,$$

where  $\tilde{c}_n(t)$  is the coefficient of  $z^n$  in the polynomial  $P_t(z)H(z)Q(z) - Q_t(z)H(z)P(z)$ , and  $|H(z)Q(z)Q_t(z)|_K \neq 0$ .

(2) If  $\hat{f}(z) = \infty$ , then either  $z = \infty \notin \{H_f = 0\}$  and  $\deg P > \deg Q$ , which implies  $\deg P_t > \deg Q_t$ , or  $Q(z) = 0$ . If  $Q(z) = 0$ , then, as  $t \rightarrow 0$ ,  $Q_t(z) \rightarrow H_f(z)Q(z) = 0$  and  $P_t(z) \rightarrow H(z)P(z) \neq 0$ . Hence  $f_t(z) \rightarrow \infty$  as  $t \rightarrow 0$ . So we have, as  $t \rightarrow 0$ ,

$$\Delta(f_t(z), \hat{f}(z)) = \frac{1}{\max\{1, |f_t(z)|_K\}} \rightarrow 0.$$

□

**Corollary 3.4.** *Let  $\{f_t\}$  be an analytic family of degree  $d \geq 2$  rational maps. If  $\deg_0 \rho_{\mathbb{L}}(\mathbf{f}) \geq 2$ , then  $\{M_t = z\}$  is a rescaling for  $\{f_t\}$  with corresponding rescaling limit  $\rho_{\mathbb{L}}(\mathbf{f})$ .*

*Proof.* Note as  $t \rightarrow 0$  there is a homogeneous polynomial  $H \in K[X, Y]$  such that  $f_t$  converges to  $H\rho_{\mathbb{L}}(\mathbf{f})$  in  $\mathbb{P}^{2d+1}(K)$ . The conclusion then follows Proposition 3.3. □

The converse of Proposition 3.3 is also true.

**Proposition 3.5.** *Let  $\{f_t\}$  be an analytic family of degree  $d \geq 1$  rational map and let  $S \subset \mathbb{P}^1(K)$  be a finite subset. Suppose  $f_t$  converges to  $\hat{f}$  pointwise, as  $t \rightarrow 0$ , on  $\mathbb{P}^1(K) \setminus S$ . Then there exists a homogeneous polynomial  $H$  of degree  $d - \deg \hat{f}$  with zeros in  $S$  such that  $f_t \rightarrow H\hat{f}$ , as  $t \rightarrow 0$ , in  $\mathbb{P}^{2d+1}(K)$ .*

*Proof.* Let  $\mathbf{f}$  be the associated rational map of  $\{f_t\}$ . Then there exists homogeneous polynomial  $H$  such that  $f_t(z)$  converges to  $H\rho_{\mathbb{L}}(\mathbf{f})$ , as  $t \rightarrow 0$ , in  $\mathbb{P}^{2d+1}(K)$ . Thus, by Proposition 3.3,  $\hat{f} = \rho_{\mathbb{L}}(\mathbf{f})$ . It is easy to check  $H$  satisfies the required conditions. □

**Corollary 3.6.** *Suppose  $K$  has positive characteristic  $p > 0$ . Let  $\{f_t(z)\} \subset K(z)$  be an analytic family of rational maps of nontrivial degree at least 2. Let  $\mathbf{f}$  be the associated rational map of  $\{f_t\}$ . If  $\mathbf{f}$  is inseparable, then all the rescaling limits of  $\{f_t\}$  are inseparable.*



*Proof.* Let  $g$  be a rescaling limit for  $\{f_t\}$ . Then by Definition 2.4 and Proposition 3.5, there exist a rescaling  $\{M_t\}$ , an integer  $q \geq 1$  and homogeneous polynomial  $H$  such that  $M_t^{-1} \circ f_t^q \circ M_t \rightarrow Hg$ . Note the associated rational map  $\mathbf{M}^{-1} \circ \mathbf{f}^q \circ \mathbf{M}$  of  $\{M_t^{-1} \circ f_t^q \circ M_t\}$  is inseparable since  $\mathbf{f}$  is inseparable. Considering the coefficients of  $g$ , we have the map  $g$  is inseparable.  $\square$

#### 4. BERKOVICH DYNAMICS

In this section, we first summarize the properties of the dynamics on a Berkovich space, see [3, 4, 13, 18]. Then we restate the results in [16], which are proven for a holomorphic family of rational maps over  $\mathbb{C}$ . These results are still true for an analytic family  $\{f_t(z)\}$  of rational maps over field  $K$  with characteristic zero.

Recall that the Berkovich Julia set  $J_{Ber}(\phi)$  of a rational map  $\phi : \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow \mathbb{P}_{Ber}^1(\mathbb{L})$  is the set consisting of all points  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L})$  such that  $\cup_{n \geq 0} \phi^n(U)$  omits finitely many points of  $\mathbb{P}_{Ber}^1(\mathbb{L})$  for any neighborhood  $U$  of  $\xi$ . The classical Julia set  $J_I(\phi)$  is  $J_{Ber}(\phi) \cap \mathbb{P}^1(\mathbb{L})$ . Let  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L}) \setminus \mathbb{P}^1(\mathbb{L})$  be a periodic point of  $\phi$  of period  $n \geq 1$ . The multiplier  $\lambda$  of  $\xi$  is defined by the local degree of  $\phi^n$  at  $\xi$ , that is,  $\lambda := \deg_\xi(\phi^n)$ . If  $|\lambda| \geq 2$ , we say  $\xi$  is repelling. If a periodic point  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L}) \setminus \mathbb{P}^1(\mathbb{L})$  is repelling, then  $\xi$  is a type II point. Let  $\mathcal{P} \subset \mathbb{P}_{Ber}^1(\mathbb{L})$  be a  $n$ -cycle of  $\phi$ . The basin of  $\mathcal{P}$  is the interior of the set of points  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L})$  such that, for all neighborhoods  $U$  of  $\mathcal{P}$ , the orbit of  $\xi$  is eventually contained in  $U$ .

Recall the tangent space  $T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$  is the set of all directions at  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L})$ . Let  $\phi : \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow \mathbb{P}_{Ber}^1(\mathbb{L})$  be a rational map. Then for any  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$ , there is a unique  $\vec{w} \in T_{\phi(\xi)} \mathbb{P}_{Ber}^1(\mathbb{L})$  such that for any  $\xi'$  sufficiently near  $\xi$ ,  $\phi(\xi') \in \mathbf{B}_{\phi(\xi)}(\vec{w})^-$ . Thus the rational map  $\phi$  induces a map  $\phi_* : T_\xi \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow T_{\phi(\xi)} \mathbb{P}_{Ber}^1(\mathbb{L})$  sending the direction  $\vec{v}$  to the corresponding direction  $\vec{w}$ .

**Proposition 4.1.** [3, Corollary 9.25, Theorem 9.26, Corollary 9.27, Proposition 9.41] *Let  $\phi : \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow \mathbb{P}_{Ber}^1(\mathbb{L})$  be a rational map of degree at least 1. Then  $\phi(\xi_G) = \xi_G$  if and only if  $\deg \rho_{\mathbb{L}}(\phi) \geq 1$ . Moreover,*

- (1) *Assume  $\phi(\xi_G) = \xi_G$ . With identifying  $T_{\xi_G} \mathbb{P}_{Ber}^1(\mathbb{L})$  to  $\mathbb{P}^1(K)$ , the following hold:*
  - (a)  $\deg_{\xi_G} \phi = \deg \rho_{\mathbb{L}}(\phi)$ ,
  - (b) *at the Gauss point  $\xi_G$ ,  $\phi_* = \rho_{\mathbb{L}}(\phi)$  on  $\mathbb{P}^1(K)$ .*
- (2) *For  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L})$  and  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$ , the image  $\phi(\mathbf{B}_\xi(\vec{v})^-)$  always contains  $\mathbf{B}_{\phi(\xi)}(\phi_* \vec{v})^-$ , and either  $\phi(\mathbf{B}_\xi(\vec{v})^-) = \mathbf{B}_{\phi(\xi)}(\phi_* \vec{v})^-$  or  $\phi(\mathbf{B}_\xi(\vec{v})^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$ . There exists an integer  $m \geq 1$  such that*
  - (a) *if  $\phi(\mathbf{B}_\xi(\vec{v})^-) = \mathbf{B}_{\phi(\xi)}(\phi_* \vec{v})^-$ , then each  $\zeta \in \mathbf{B}_{\phi(\xi)}(\phi_* \vec{v})^-$  has  $m$  preimages in  $\mathbf{B}_\xi(\vec{v})^-$ , counting multiplicities;*
  - (b) *if  $\phi(\mathbf{B}_\xi(\vec{v})^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$ , there is an integer  $N \geq m$  such that each  $\zeta \in \mathbf{B}_{\phi(\xi)}(\phi_* \vec{v})^-$  has  $N$  preimages in  $\mathbf{B}_\xi(\vec{v})^-$  and each  $\zeta \in \mathbb{P}_{Ber}^1(\mathbb{L}) \setminus \mathbf{B}_{\phi(\xi)}(\phi_* \vec{v})^-$  has  $N - m$  preimages in  $\mathbf{B}_\xi(\vec{v})^-$ , counting multiplicities.*

Based on Proposition 3.3 and Proposition 4.1, we have

**Proposition 4.2.** [16, Proposition 3.4, Lemma 3.6, Lemma 3.7] *Let  $\{f_t(z)\} \subset K(z)$  be an analytic family of rational maps of nontrivial degree at least 2, and let  $\{M_t\}$  and  $\{L_t\}$  be moving frames. Denote by  $\mathbf{f}$ ,  $\mathbf{M}$  and  $\mathbf{L}$  the associated rational maps. Then*

- (1) *For all  $l \geq 1$ , the following are equivalent:*

- (a) *There exists a rational map  $g : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$  of degree at least  $d \geq 1$  such that  $M_t^{-1} \circ f_t^l \circ M_t$  converges to  $g$  pointwise, as  $t \rightarrow 0$ , on  $\mathbb{P}^1(K)$  off a finite subset.*
  - (b)  *$\mathbf{f}^l(\xi) = \xi$ , where  $\xi = \mathbf{M}(\xi_G)$  and  $\deg_\xi \mathbf{f}^l = d$ .*
- In the case in which (a) and (b) hold, the map  $(\mathbf{f}^l)_* : T_\xi \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$  is conjugate via a  $M \in \text{Rat}_1(K)$  to  $g : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$ .*
- (2) *Moving frames  $\{M_t\}$  and  $\{L_t\}$  are equivalent if and only if  $\mathbf{M}(\xi_G) = \mathbf{L}(\xi_G)$ .*
  - (3) *The following are equivalent:*
    - (a)  *$\mathbf{f} \circ \mathbf{M}(\xi_G) = \mathbf{L}(\xi_G)$ .*
    - (b) *As  $t \rightarrow 0$ ,  $L_t^{-1} \circ f_t \circ M_t$  converges to some nonconstant rational map  $g : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$  pointwise outside some finite subset.*

**Corollary 4.3.** *Let  $\{f_t\} \subset K(z)$  be an analytic family of degree at least 2 rational maps. Suppose  $\{f_t\}$  has (potentially) good reduction. Then there is at most one rescaling, up to equivalence, for  $\{f_t\}$ , and this rescaling is of period 1.*

*Proof.* Let  $\mathbf{f}$  be the associated rational map of  $\{f_t\}$ . Then  $\mathbf{f}$  has (potentially) good reduction. Then the classical Julia set  $\mathcal{J}_I(\mathbf{f}) = \emptyset$  [4, Proposition 4.22] and the Berkovich Julia set  $\mathcal{J}_{Ber}(\mathbf{f})$  is a singleton set [4, Corollary 6.25]. Thus  $\phi$  has no repelling periodic points of type I and has only one repelling periodic point since  $\mathcal{J}_{Ber}(\mathbf{f})$  is contained in the closure of the set of repelling periodic points of  $\mathbf{f}$  in  $\mathbb{P}_{Ber}^1(\mathbb{L})$  [4, Theorem 7.27]. By Proposition 4.2, all the rescalings of  $\{f_t\}$  are equivalent and they are of period 1.  $\square$

To relate the critical points of  $\mathbf{f}$  and the rescaling limits of  $\{f_t\}$ , we first state the following non-Archimedean Rolle's theorem:

**Lemma 4.4.** *[11, Application 1] Suppose  $K$  has characteristic zero and residue characteristic zero. Let  $\phi \in K(z)$  be a rational map of degree at least 1. If  $\phi$  has two distinct zeros in the closed disk  $\overline{D}(a, r)$ , then it has a critical point in  $\overline{D}(a, r)$ .*

We should mention here Lemma 4.4 is not true in general. If  $K$  has characteristic zero and residue characteristic  $p > 0$ , then under same assumptions,  $\phi$  is only guaranteed to have a critical point in  $\overline{D}(a, r|p|_K|^{-1/(p-1)})$  which are strictly larger than  $\overline{D}(a, r)$ . If  $K$  has characteristic  $p > 0$ , consider the field  $\mathbb{L}$  and  $\phi(z) = z^p - z \in \mathbb{L}(z)$ . Then  $\phi$  has only one critical point, which is  $\infty \in \mathbb{P}^1(\mathbb{L})$ . However,  $\phi$  has  $p$  zeros in  $\overline{D}(0, 1) \subset \mathbb{L}$ . For more details about rational maps with one critical point, we refer [12].

Applying the non-Archimedean Rolle's theorem and using the same proof in [16], we have

**Proposition 4.5.** *Suppose  $K$  has characteristic zero. Consider a rational map  $\phi : \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow \mathbb{P}_{Ber}^1(\mathbb{L})$  of degree at least 2. Let  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L})$  be a type II point and let  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$ . If  $\phi : \mathbf{B}_\xi(\vec{v})^- \rightarrow \phi(\mathbf{B}_\xi(\vec{v})^-)$  is not injective, then there is a critical point of  $\phi$  in  $\mathbf{B}_\xi(\vec{v})^-$  such that the corresponding critical value  $\phi(c) \in \mathbf{B}_{\phi(\xi)}(\phi_*(\vec{v}))^-$ .*

**Proposition 4.6.** *Suppose  $K$  has characteristic zero. Consider a rational map  $\phi : \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow \mathbb{P}_{Ber}^1(\mathbb{L})$  of degree at least 2. Let  $\mathcal{P}$  be a type II periodic orbit of period  $q \geq 1$  of  $\phi$ . Assume the basin of  $\mathcal{P}$  is free of critical points of  $\phi$ . Then, for all  $\xi \in \mathcal{P}$ , every  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$  with  $\phi^q(\mathbf{B}_\xi(\vec{v})^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$  has a finite forward orbit under  $(\phi^q)_*$ . Moreover, if  $\deg(\phi^q)_* \geq 2$ , then  $(\phi^q)_*$  is postcritically finite.*

In the next section, we establish analogs of these two propositions in positive characteristic, and from this deduce our main result.

## 5. RATIONAL MAPS OVER $K$ WITH POSITIVE CHARACTERISTIC

Assume that the field  $K$  has positive characteristic  $p > 0$ . A nonconstant rational map  $\phi \in K(z)$  can then be written as  $\phi(z) = \phi_1(z^{p^j})$  for some integer  $j \geq 0$ , where  $\phi_1$  is separable. Recall the ramification locus  $\mathcal{R}_\phi = \{\xi \in \mathbb{P}_{Ber}^1(K) : \deg_\xi \phi \geq 2\}$  and  $\phi$  is tame if  $\mathcal{R}_\phi$  contains finitely many points with valence at least 3. We say a rational map  $\phi \in K(z)$  is tamely ramified if the characteristic of  $K$  does not divide the multiplicity  $m_\phi(z)$  for any  $z \in \mathbb{P}^1(K)$ . The space  $\mathbb{P}_{Ber}^1(K) \setminus \mathbb{P}^1(K)$  carries a natural metrizable topology, the strong topology, see [3, 10]. With respect to this metric, there exists  $r > 0$  such that the ramification locus  $\mathcal{R}_\phi$  is in an  $r$ -neighborhood of the connected hull  $Hull(Crit(\phi))$  of critical set if and only if  $\phi$  is tamely ramified [11, Theorem E]. If  $\phi$  is separable, the extreme case  $\mathcal{R}_\phi \subseteq Hull(Crit(\phi))$  is equivalent to  $\phi$  is tame [10, Corollary 7.13].

We can prove the following Non-Archimedean Rolle's theorem for a separable tame rational map over a field  $K$  with positive characteristic.

**Lemma 5.1.** *Suppose  $K$  has positive characteristic  $p > 0$ . Let  $\phi \in K(z)$  be a separable tame rational map of degree at least 1. If  $\phi$  has two distinct zeros in the closed disk  $\overline{D}(a, r)$ , then it has a critical point in  $\overline{D}(a, r)$ .*

*Proof.* Suppose there is no critical point in  $\overline{D}(a, r)$ . Let  $\xi_{a,r} \in \mathbb{P}_{Ber}^1(K)$  be the point corresponding to the closed disk  $\overline{D}(a, r)$ . Then  $\xi_{a,r} \notin Hull(Crit(\phi))$ . Let  $\vec{v} \in T_{\xi_{a,r}} \mathbb{P}_{Ber}^1(K)$  be the direction such that  $\overline{D}(a, r) \subset \mathbb{P}_{Ber}^1(K) \setminus \mathbf{B}_{\xi_{a,r}}(\vec{v})^-$ . Then the set  $\mathbb{P}_{Ber}^1(K) \setminus \mathbf{B}_{\xi_{a,r}}(\vec{v})^-$  is disjoint with  $Hull(Crit(\phi))$ . So  $\mathbb{P}_{Ber}^1(K) \setminus \mathbf{B}_{\xi_{a,r}}(\vec{v})^- \cap \mathcal{R}_\phi = \emptyset$ . Since  $\mathcal{R}_\phi$  is closed, there exist  $\xi \in \mathbb{P}_{Ber}^1(K)$  and  $\vec{w} \in T_\xi \mathbb{P}_{Ber}^1(K)$  such that  $\mathbb{P}_{Ber}^1(K) \setminus \mathbf{B}_{\xi_{a,r}}(\vec{v})^- \subset \mathbf{B}_\xi(\vec{w})^-$  and  $\mathbf{B}_\xi(\vec{w})^- \cap \mathcal{R}_\phi = \emptyset$ . Hence  $\phi$  is injective on  $\mathbf{B}_\xi(\vec{w})^-$  [10, Corollary 3.8]. So  $\phi$  is injective on  $\mathbb{P}_{Ber}^1(K) \setminus \mathbf{B}_{\xi_{a,r}}(\vec{v})^-$ . Thus  $\phi$  is injective on the closed disk  $\overline{D}(a, r)$ . So  $\phi$  has at most one zero in  $\overline{D}(a, r)$ . It is a contradiction.  $\square$

Applying Lemma 5.1 and the argument in [16, Lemma 4.2], we obtain an analog of Proposition 4.5:

**Proposition 5.2.** *Suppose  $K$  has positive characteristic  $p > 0$  and consider a separable tame rational map  $\phi : \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow \mathbb{P}_{Ber}^1(\mathbb{L})$  of degree at least 2. Let  $\xi \in \mathbb{P}_{Ber}^1(\mathbb{L})$  be a type II point and let  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$ . If  $\phi : \mathbf{B}_\xi(\vec{v})^- \rightarrow \phi(\mathbf{B}_\xi(\vec{v})^-)$  is not injective, then there is a critical point of  $\phi$  in  $\mathbf{B}_\xi(\vec{v})^-$  such that the corresponding critical value  $\phi(c) \in \mathbf{B}_{\phi(\xi)}(\phi_*(\vec{v}))^-$ .*

We now prove an analogy of Proposition 4.6:

**Proposition 5.3.** *Suppose  $K$  has positive characteristic  $p > 0$ . Consider a rational map  $\phi : \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow \mathbb{P}_{Ber}^1(\mathbb{L})$  of nontrivial degree at least 2 and suppose the separable part  $\phi_1$  of  $\phi$  is tame. Let  $\mathcal{P}$  be a type II periodic orbit of period  $q \geq 1$  of  $\phi$ . Assume the basin of  $\mathcal{P}$  is free of critical values of  $\phi_1$ . Then, for all  $\xi \in \mathcal{P}$ , every  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$  with  $\phi^q(\mathbf{B}_\xi(\vec{v})^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$  has a finite forward orbit under  $(\phi^q)_*$ .*

*Proof.* Let  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$  such that  $\phi^q(\mathbf{B}_\xi(\vec{v})^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$  and  $\vec{v}$  has an infinite forward orbit under  $(\phi^q)_*$ . We will show there exists a critical value of  $\phi_1$  in

the basin of  $\mathcal{P}$ . Let  $q_0 \geq 1$  be the smallest integer such that  $\phi_1(\mathbf{B}_{\phi_2 \circ \phi^{q_0}(\xi)}((\phi_2 \circ \phi^{q_0})_*(\vec{v}))^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$ . Then by Proposition 4.1 and Proposition 5.2, there is a critical point  $c \in \text{Crit}(\phi_1)$  such that  $\phi_1(c) \in \mathbf{B}_{\phi^{q_0+1}(\xi)}((\phi^{q_0+1})_*(\vec{v}))^-$ . Now we show for each  $n \geq q_0 + 1$ ,  $\mathbf{B}_{\phi^n(\xi)}((\phi^n)_*(\vec{v}))^-$  contains a point in the forward orbit of a critical value of  $\phi_1$ . By induction, suppose it holds for  $n = k \geq q_0 + 1$ . If  $\phi(\mathbf{B}_{\phi^k(\xi)}((\phi^k)_*(\vec{v}))^-) = \mathbf{B}_{\phi^{k+1}(\xi)}((\phi^{k+1})_*(\vec{v}))^-$ , then  $\mathbf{B}_{\phi^{k+1}(\xi)}((\phi^{k+1})_*(\vec{v}))^-$  contains a point in the forward orbit of a critical value of  $\phi_1$ . If  $\phi(\mathbf{B}_{\phi^k(\xi)}((\phi^k)_*(\vec{v}))^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$ , then  $\phi_1(\mathbf{B}_{\phi_2 \circ \phi^k(\xi)}((\phi_2 \circ \phi^k)_*(\vec{v}))^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$ . By Proposition 4.1 and Proposition 5.2,  $\mathbf{B}_{\phi^{k+1}(\xi)}((\phi^{k+1})_*(\vec{v}))^-$  contains a critical value of  $\phi_1$ .

Thus, for  $n$  large,  $\mathbf{B}_{\phi^n(\xi)}((\phi^n)_*(\vec{v}))^-$  contains a point in the forward orbit of a critical value of  $\phi_1$ . Note for  $n$  sufficiently large, say  $n \geq n_0$ ,  $\phi(\mathbf{B}_{\phi^n(\xi)}((\phi^n)_*(\vec{v}))^-) \neq \mathbb{P}_{Ber}^1(\mathbb{L})$ . Suppose  $\phi^l(\phi_1(c)) \in \mathbf{B}_{\phi^{n_0+q}(\xi)}((\phi^{n_0+q})_*(\vec{v}))^-$  for some  $c \in \text{Crit}(\phi_1)$  and  $l \geq 0$ , then  $\phi^{n_0+l}(\phi_1(c)) \rightarrow \xi$ , as  $n \rightarrow \infty$ , in the Gelfand topology. Thus  $\phi_1(c)$  is in the basin of the periodic cycle  $\mathcal{P}$ .  $\square$

**Corollary 5.4.** *Under the same assumptions in Proposition 5.3, if  $\deg_0(\phi^q)_* \geq 2$ , then  $(\phi^q)_*$  is postcritically finite at the nontrivial critical points.*

*Proof.* Suppose  $(\phi^q)_*$  is not postcritically finite at the nontrivial critical points. Let  $\vec{v} \in T_\xi \mathbb{P}_{Ber}^1(\mathbb{L})$  be a nontrivial critical point of  $(\phi^q)_*$  with infinite forward orbit, then there exists  $j \geq 0$  such that  $(\phi_2 \circ \phi^j)^{-1}(\text{Crit}(\phi_1)) \cap \mathbf{B}_\xi(\vec{v})^- \neq \emptyset$ . If  $\phi^n(\mathbf{B}_\xi(\vec{v})^-) \neq \mathbb{P}_{Ber}^1(\mathbb{L})$  for all  $n \geq 1$ , then

$$\phi_1(\text{Crit}(\phi_1)) \cap \mathbf{B}_{\phi^{j+1}(\xi)}((\phi^{j+1})_*(\vec{v}))^- \neq \emptyset.$$

Hence for all  $n \geq 0$ ,  $\phi^n(\phi_1(\text{Crit}(\phi_1))) \cap \mathbf{B}_{\phi^{n+j+1}(\xi)}((\phi^{n+j+1})_*(\vec{v}))^- \neq \emptyset$ . So there exists  $c \in \text{Crit}(\phi_1)$  such that  $\phi_1(c)$  in the basin of  $\mathcal{P}$ . If there exists  $n_0 \geq 1$  such that  $\phi^{n_0}(\mathbf{B}_\xi(\vec{v})^-) = \mathbb{P}_{Ber}^1(\mathbb{L})$ , then  $\vec{v}$  has a finite forward orbit by Proposition 5.3.  $\square$

Based on Proposition 4.6 and Corollary 5.4, applying the argument in [16], we can prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $\{M_t^{(1)}\}, \dots, \{M_t^{(n)}\}$  be pairwise dynamically independent rescalings for  $\{f_t\}$  of periods  $q_1, \dots, q_n$  such that the corresponding rescaling limits are not postcritically finite at nontrivial critical points. Let  $\mathbf{f}$  be the associated rational map of  $\{f_t\}$  and  $\mathbf{M}^{(1)}, \dots, \mathbf{M}^{(n)}$  be the associated rational maps of  $\{M_t^{(1)}\}, \dots, \{M_t^{(n)}\}$ . Let  $\xi_j = \mathbf{M}^{(j)}(\xi_G) \in \mathbb{P}_{Ber}^1(\mathbb{L})$  for  $j = 1, \dots, n$ . Then by Proposition 4.2, for all  $j = 1, \dots, n$ ,  $\mathbf{f}^{q_j} : T_{\xi_j} \mathbb{P}_{Ber}^1(\mathbb{L}) \rightarrow T_{\xi_j} \mathbb{P}_{Ber}^1(\mathbb{L})$  is not postcritically finite at the nontrivial critical points, and the points  $\xi_1, \dots, \xi_n$  are in pairwise distinct periodic orbits of  $\mathbf{f}$ . Note the separable part of  $\mathbf{f}$  has at most  $2\deg_0 \mathbf{f} - 2$  critical points, hence it has at most  $2\deg_0 \mathbf{f} - 2$  critical values. Then by Proposition 4.6 for the case  $\text{char } K = 0$  and Corollary 5.4 for the case  $\text{char } K > 0$ , we have  $n \leq 2\deg_0 \mathbf{f} - 2$ .  $\square$

## 6. EXAMPLES

In this section, we give some examples to illustrate rescaling limits in non-Archimedean dynamics. We refer [16] for more examples. All examples in [16] are holomorphic families over  $\mathbb{C}$ , which can be considered as analytic families over non-Archimedean fields.

Now let  $p > 0$  be a prime number. Denote by  $K$  the completion of the formal Puiseux series  $\overline{\mathbb{F}}_p\{\{s\}\}$  over  $\overline{\mathbb{F}}_p$  with respect to its nontrivial non-Archimedean absolute value. Then  $\text{Char } K = p > 0$ .

**Example 6.1.** *Polynomials with quadratic separable parts.*

Given sufficiently small  $t \in K \setminus \{0\}$ . Consider the map

$$f_t(z) = (z^p - (s+t))(z^p - s) \in K(z).$$

Then associated map  $\mathbf{f}$  of  $\{f_t\}$  has separable part  $\mathbf{f}_1(z) = (z - (s+t))(z - s)$ . Note that  $\mathbf{f}_1$  is tame if and only if  $p \geq 3$ . In fact, if  $p = 2$ ,  $\mathbf{f}_1$  has only one critical point at  $\infty$ .

Note  $\{f_t\}$  is an analytic family that has good reduction. Then, up to equivalence, the moving frame  $\{M_t(z) = z\}$  is the unique possible rescaling. The corresponding limit is  $g(z) = (z^p - s)^2$ . If  $p = 2$ , then  $g(z) = z^4 + s^2$  has nontrivial degree 1. If  $p \geq 3$ ,  $g(z) = (z^p - s)^2$  has separable part  $g_1(z) = (z - s)^2$ . Note  $\text{Crit}(g_1) = \{s, \infty\}$  and  $g_1(s)$  has an infinite forward orbit under  $g$ . Thus  $g$  is a rescaling limit that is not postcritically finite at nontrivial critical points.

**Example 6.2.** *Connected Julia sets.*

Let  $a \in K$  with  $0 < |a|_K < 1$  and  $b \in K$  with  $|b|_K = |b - 1|_K = 1$ . Define

$$\phi_{a,b}(z) = \frac{az^6 + 1}{az^6 + z(z-1)(z-b)} \in K(z).$$

Then the Berkovich Julia set  $J_{\text{Ber}}(\phi_{a,b})$  is connected but not contained in a line segment [2].

For sufficiently small  $t \in K \setminus \{0\}$ , let  $a = t^q$ , where  $q = 3p^2 + 1$ , and fix  $b \in K$ . Define  $f_t(z) = \phi_{t^q,b}(z^p)$ . Then  $\{f_t\}$  is a degenerated analytic family defined in a neighborhood of  $t = 0$ . Let  $\mathbf{f}$  be the associated map of  $\{f_t\}$ . Then the separable part  $\mathbf{f}_1$  of  $\mathbf{f}$  is  $\mathbf{f}_1(z) = \frac{t^q z^6 + 1}{t^q z^6 + z(z-1)(z-b)}$ , which has a connected Berkovich Julia set  $J_{\text{Ber}}(\mathbf{f}_1)$ .

First the moving frame  $\{M_t(z) = z\}$  is a rescaling of period 1 with rescaling limit

$$g(z) = \frac{1}{z(z-1)(z-b)} \circ z^p.$$

Note  $\mathbf{f}_1$  maps the segment  $[\xi_{0,|t|^{q/2}}, \xi_G]$  isometrically onto the segment  $[\xi_{0,|t|^{-q/2}}, \xi_G]$ . And  $\mathbf{f}_1$  maps the segment  $[\xi_G, \xi_{0,|t|^{-q/6}}]$  bijectively to the segment  $[\xi_G, \xi_{0,|t|^{q/2}}]$  and the segment  $[\xi_{0,|t|^{-q/6}}, \xi_{0,|t|^{-q/3}}]$  bijectively to the segment  $[\xi_{0,|t|^{q/2}}, \xi_G]$ , stretching by a factor of 3, respectively. We may expect there exists a point  $\xi_{0,r} \in \mathbb{P}_{\text{Ber}}^1(\mathbb{L})$  such that  $\mathbf{f}^2(\xi_{0,r}) = \xi_{0,r}$ . Indeed, we can choose  $r = |t|_{\mathbb{L}}$ . Let  $L_t(z) = tz$ . Then the moving frame  $\{L_t(z)\}$  is a rescaling of period 2 leading to rescaling limit

$$h(z) = \begin{cases} \frac{1}{b^6 z^3} \circ z^4, & \text{if } p = 2, \\ -\frac{1}{b^9 z} \circ z^{27}, & \text{if } p = 3, \\ -\frac{1}{b^{3p} z^3} \circ z^{p^2}, & \text{if } p \geq 5. \end{cases}$$

**Example 6.3.** *McMullen maps.*

This example is an analog of [16, 2.4]. Given sufficiently small  $t \in K \setminus \{0\}$ , consider the map

$$f_t(z) = z^{2p} + \frac{t^{1+2p^2}}{z^p} \in K(z).$$

Then  $\{f_t\}$  is a degenerate analytic family defined in a neighborhood of  $t = 0$ . The associated map  $\mathbf{f}$  of  $\{f_t\}$  has separable part  $\mathbf{f}_1(z) = z^2 + \frac{t^{1+2p^2}}{z}$ . By [10, Corollary 6.6], when  $p \geq 5$ , the map  $\mathbf{f}_1(z)$  is tame. In fact, when  $p = 3$ , at every type II point,  $\phi$  has separable reduction. Then by [10, Corollary 7.13],  $\phi$  is tame. When  $p = 2$ ,  $\phi$  has a unique critical point. Thus,  $\phi$  is tame if and only if  $p \geq 3$ .

Obviously, the moving frame  $\{M_t(z) = z\}$  is a rescaling of period 1. The corresponding rescaling limit is  $g(z) = z^{2p}$ , which has a tame separable part  $g_1(z) = z^2$ . Note the nontrivial critical set  $\text{Crit}_0(g) = \{0, \infty\}$ . So the rescaling limit  $g$  is postcritically finite at the nontrivial critical points.

Moreover, the moving frame  $\{L_t(z) = tz\}$  is a rescaling of period 2 for  $\{f_t\}$ , which leads to the rescaling limit  $h(z) = z^{-2p^2}$ . The rescaling limit  $h(z)$  is also postcritically finite at the nontrivial critical points.

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